

A LEXICOGRAPHIC SHELLABILITY CHARACTERIZATION OF GEOMETRIC LATTICES

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ABSTRACT. Geometric lattices are characterized as those finite, atomic lattices such that every atom ordering induces a lexicographic shelling given by an edge labeling known as a minimal labeling. Geometric lattices arise as the intersection lattices of central hyperplane arrangements and more generally as the lattices of flats for matroids. This result fits into a similar paradigm as McNamara's 2003 characterization of supersolvable lattices as those lattices admitting other types of lexicographic shellings, namely the so-called S_n - EL -labelings.

1. INTRODUCTION

In [6], McNamara proved that supersolvable lattices can be characterized as lattices admitting a certain type of EL -labeling known as an S_n - EL -labeling. Each maximal chain is labeled by the set of labels $\{1, \dots, n\}$ with each label used exactly once in each maximal chain. Previously, Stanley had proven that all supersolvable lattices admit such EL -labelings in [9]. Thus, McNamara's result gave a new characterization of supersolvable lattices: that a finite lattice is supersolvable if and only if it has an S_n - EL -labeling.

We now give a result of a similar spirit for geometric lattices – a new characterization of geometric lattices as the lattices admitting a family of lexicographic shellings induced by the various possible orderings on the atoms. Geometric lattices are well known to have the property that every atom ordering induces an EL -labeling by labeling each covering relation $u \prec v$ with the smallest atom that is below v but not u . We prove that this is actually a characterization of geometric lattices, i.e. that all finite atomic lattices in which every atom order induces an EL -labeling are in fact geometric lattices.

There is an extensive literature surrounding the notion of lexicographic shellability. Numerous important classes of partially ordered sets have been proven to admit various types of lexicographic shellings.

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For example, distributive lattices (Björner [2]), upper semimodular lattices (Garsia [5]), geometric lattices (Stanley [10], Björner [2]) and semilattices (Wachs-Walker [12]), supersolvable lattices (Stanley [9]), subgroup lattices of solvable groups (Shareshian [8], Woodroffe [13]), and Bruhat order (Björner-Wachs [4]) are all known to be shellable.

Our aim is to take things in a new direction, namely to initiate a study of which classes of posets can actually be characterized in terms of the type of lexicographic shellings they have. McNamara's work was a first such result. Our work shows that this was not an isolated phenomenon, but in fact may have been one of the first indications of the power of lexicographic shellability as a tool for characterizing different classes of posets. Important classes of finite groups (e.g. supersolvable groups and solvable groups) have already been characterized in terms of shellability properties of their posets of subgroups (see results of Stanley [9], Shareshian [8], and Woodroffe [13]), but this possibility seems to extend well beyond lattices of subgroups.

One motivation for lexicographic shellability is as a tool to compute Möbius functions and thereby solve counting problems by inclusion-exclusion. Geometric lattices arise as the intersection lattices of real, central hyperplane arrangements. For example, the partition lattice is the intersection lattice of the type A Coxeter arrangement. Zaslavsky expressed the number of regions in the complement of a real hyperplane arrangement in terms of Möbius functions of geometric lattices and semi-lattices in [14], another reason for particular interest in the Möbius functions, and hence the shellability, of geometric lattices.

McNamara's characterization of supersolvable lattices has given a useful new way of proving lattices to be supersolvable. It has already been used this way in [1]. Our result creates the same potential for geometric lattices.

2. BACKGROUND AND STATEMENT OF MAIN RESULT

Let P be a finite poset. Let $E(P)$ denote the set of edges of the Hasse diagram of P . We write $x \lessdot y$ or $x \prec y$ to indicate that y covers x in P , namely $x \leq z \leq y$ implies $z = x$ or $z = y$. If $\lambda : E(P) \rightarrow \mathbb{N}$ is an edge labeling of the Hasse diagram of P and $x \lessdot y$, then we write $\lambda(x, y)$ to indicate the label given to the edge from x to y . Recall that λ is an *EL-labeling* for P if for every interval $[x, y] \subset P$, there is a unique *rising chain* $C := x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_j = y$ where $\lambda(x, x_2) \leq \lambda(x_2, x_3) \leq \cdots \leq \lambda(x_{j-1}, y)$, and the label sequence of C is lexicographically smaller than the label sequence of every other

saturated chain in the interval $[x, y]$. It is well known that an *EL*-labeling gives a shelling order for the facets (maximal faces) of the order complex $\Delta(P)$ of P . *EL*-labelings have been constructed for a remarkably large and diverse assortment of important classes of posets.

We now review the notion of geometric lattice as well as the types of *EL*-labelings which they are already known to possess. An *atom* in a poset P with unique minimal element $\hat{0}$ is any $a \in P$ such that $\hat{0} \prec a$. A *lattice* is a poset such that any pair of elements x, y has a unique least upper bound $x \vee y$ and a unique greatest lower bound $x \wedge y$. A lattice is *atomic* if each element is a join of atoms. A lattice is *semimodular* if it has a rank function ρ that satisfying

$$(i) \quad \rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y).$$

A finite lattice is *geometric* if it is atomic and semimodular. Our main goal is to give a new characterization of geometric lattices.

Next we introduce the relevant family of edge-labelings. Let L be a finite atomic lattice with n atoms. Let $\mathcal{A}(L)$ denote the set of atoms of L . For $x \in L$, let

$$A(x) = \{a \leq x \mid a \in \mathcal{A}(L)\}.$$

Further, given a bijection $\gamma : \mathcal{A}(L) \rightarrow [n]$, let $\gamma(A(x))$ denote the set $\{\gamma(a) \mid a \in A(x)\}$. Then the map γ induces a *minimal labeling* $\lambda_\gamma : E(L) \rightarrow [n]$ by the rule $\lambda_\gamma(x, y) = \min\{\gamma(A(y)) \setminus \gamma(A(x))\}$.

Theorem 1 (Björner). *The minimal labeling resulting from any ordering of the atoms in a geometric lattice is an EL-labeling.*

Our interest is in using the existence of a certain family of edge labelings for a poset P to show that P fits into an important class of posets, namely the geometric lattices.

Theorem 2. *Let L be a finite atomic lattice with n atoms. If for every atom ordering, i.e. every bijective map $\gamma : \mathcal{A}(L) \rightarrow [n]$, the labeling λ_γ is an EL-labeling, then L is a geometric lattice.*

In [3], there is a result of a similar flavor (Theorem 7.3.4) concerning matroid complexes: a simplicial complex Δ is the independence complex of a matroid if and only if Δ is pure and every ordering of the vertices induces a shelling of Δ . Geometric lattices are also intricately connected to matroids. Though it seems interesting to note the strong analogy between the necessary hypotheses for Theorem 7.3.4 of [3] and those of our main theorem, a relationship we discuss next, our result is a fundamentally different result.

If $M = M(S)$ is a matroid of rank r on a finite set S , the *independence complex* $IN(M)$ is the $(r - 1)$ -dimensional simplicial complex

formed by the family of all independent sets in M . On the other hand, a geometric lattice is the lattice of *flats*, or closed sets, of a matroid. Using our terminology, the matroid structure of a geometric lattice L is the matroid with ground set $\mathcal{A}(L)$ and the closure operator $cl(W)$ on a subset $W \subset \mathcal{A}(L)$ is $cl(W) = \{a \in \mathcal{A}(L) \mid x \text{ is the join of the atoms in } W\}$.

Since a geometric lattice is a finite, semi-modular, atomic lattice, it will suffice to prove that L is graded with a rank function ρ satisfying $\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y)$, i.e. with rank function satisfying (i) above. The following proposition, which appears as Proposition 3.3.2 in [11], gives a convenient property often referred to as the ‘diamond property’. It implies not only the existence of a rank function, but also the inequality (i) above.

Proposition 3 (Stanley, Proposition 3.3.2). *Let L be a finite lattice. The following two conditions are equivalent:*

- *L is graded, and the rank function ρ of L satisfies*

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y)$$

for all $x, y \in L$.

- *If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y .*

Our basic approach for proving Theorem 2 will be to take any pair of elements x, y both covering $x \wedge y$ but not both covered by $x \vee y$ and construct an atom ordering in terms of x and y whose minimal labeling will not be an EL-labeling.

Our initial motivation was the question of whether posets admitting certain types of edge labelings were in fact more general than geometric lattices, so as to see whether it made sense to generalize results of [7] from geometric lattices to more general posets with the types of edge-labelings one has for geometric lattices. Our main result says that these two classes of atomic lattices are in fact exactly the same, i.e., the latter is not actually any larger.

3. PROOF OF MAIN RESULT

The remainder of the paper is devoted to proving Theorem 2, i.e. to proving that finite atomic lattices in which every atom ordering induces an EL-labeling are geometric lattices. To this end, we first develop some helpful properties of minimal labelings.

Lemma 4. *Let L be a finite atomic lattice and let λ_γ be a minimal labeling of $E(L)$. Then for each chain $C = x_1 \leq \dots \leq x_k$, $\lambda_\gamma(x_i, x_{i+1}) \neq \lambda_\gamma(x_j, x_{j+1})$ whenever $i \neq j$. In other words, the labels on any particular saturated chain are distinct.*

Proof. This is immediate from the fact that $A(x_{j+1}) \setminus A(x_j)$ is by definition disjoint from $A(x_{i+1}) \setminus A(x_i)$ for $i \neq j$. \square

Lemma 5. *Let L be a finite atomic lattice. If $A(u) \subseteq A(v)$, then $u \leq v$.*

Proof. Since L is atomic lattice, we can write

$$u = \bigvee_{a \in A(u)} a, \quad v = \bigvee_{b \in A(v)} b$$

So if $A(u) \subset A(v)$, we have $v = \bigvee_{a \in S} a$ where $S = A(u) \cup (A(v) \setminus A(u))$, so $v = u \vee \bigvee_{a \in A(v) \setminus A(u)} a$. \square

We will use the following statement that is equivalent to Lemma 5: if v is not less than or equal to u , then there exists $a_v \in A(v)$ such that $a_v \notin A(u)$.

Lemma 6. *Let L be a finite atomic lattice. Suppose that $x, y \in L$ both cover $x \wedge y$, but that $x \vee y$ does not cover x . Given any atom a_y such that $y = (x \wedge y) \vee a_y$, then $a_y \notin A(z)$ for all z such that $x \lessdot z$.*

Proof. Let $y = (x \wedge y) \vee a_y$, for some $a_y \in \mathcal{A}(L)$. Assume for contradiction that there exists z where $x \lessdot z$ and $a_y \in A(z)$. Since y covers $x \wedge y$, we know that y and z are not comparable. Now $a_y \in A(z)$ implies $a_y \leq z$. But then we have $a_y \leq z$ and $x \wedge y \leq z$, hence $y = (x \wedge y) \vee a_y \leq z$, a contradiction. \square

Now we are ready to prove the main theorem:

Proof of Theorem 1. Assume by way of contradiction that L and λ satisfy all of the hypotheses of Theorem 1, but the diamond property does not hold. This means there exist $x, y \in L$ such that x and y both cover $x \wedge y$ but $x \vee y$ does not cover at least one of x and y . Note this immediately implies that L has at least three atoms, because the structure of a two-atom atomic lattice must be a diamond. Assume without loss of generality that $x \vee y$ does not cover x .

Let us begin with the case where x and y are atoms, then later show how to generalize the argument. Choose any atom ordering $\gamma : \mathcal{A}(L) \rightarrow [n]$ such that $\gamma(x) = 1$ and $\gamma(y) = 2$. Since x and y are atoms, $x \wedge y = \hat{0}$. Furthermore, $A(x) = \{x\}$, $A(y) = \{y\}$, so we have $\lambda_\gamma(\hat{0}, x) = 1$ and $\lambda_\gamma(\hat{0}, y) = 2$. Let C be the lexicographically smallest chain in the interval $[\hat{0}, x \vee y]$. As 1 is the smallest label, we know C begins with the covering relation $\hat{0} \lessdot x$ and we can write $C = \hat{0} \lessdot x_0 = x \lessdot x_1 \lessdot \cdots \lessdot x_k = x \vee y$ for some x_1, \dots, x_{k-1} .

Since $x \vee y$ does not cover x , we know that $k \geq 2$. We also have that $y \notin A(x_{k-1})$, because otherwise we would have $y \in A(x_{k-1})$, implying $y \leq x_{k-1}$ and hence $x_{k-1} = x \vee y$, a contradiction. So, $y \notin A(x_{k-1})$, implying $2 \neq \lambda_\gamma(x_{k-2}, x_{k-1})$. By Lemma 4, there is no repetition in our label sequences, and 1 was already used in the label sequence as $\lambda_\gamma(\hat{0}, x)$, implying $1 \neq \lambda_\gamma(x_{k-2}, x_{k-1})$. Thus, $\lambda_\gamma(x_{k-2}, x_{k-1}) \geq 3$. But since $2 \in \gamma(A(x \vee y) \setminus A(x_{k-1}))$ and $1 \notin \gamma(A(x \vee y) \setminus A(x_{k-1}))$, we know that $2 = \min\{\gamma(A(x \vee y) \setminus A(x_{k-1}))\}$ and so $\lambda_\gamma(x_{k-2}, x_{k-1}) > \lambda_\gamma(x_{k-1}, x_k)$. Therefore, the lexicographically smallest chain in this interval $[\hat{0}, x \vee y]$ has a descent, which contradicts the fact that λ_γ is an *EL*-labeling for every γ .

Now we consider the case where x and y need not be atoms. Again, assume by way of contradiction that $x \wedge y \lessdot x$ and $x \wedge y \lessdot y$, but $x \vee y$ does not cover x . By Lemma 5, we can choose some atom $a_x \in A(x) \setminus A(x \wedge y)$ such that $a_x \notin A(y)$. Since L is an atomic lattice, there must exist $a_y \in A(y)$ such that $(x \wedge y) \vee a_y = y$. By Lemma 6, $a_y \notin A(z)$ for any z such that $x \lessdot z$. Clearly $a_y \neq a_x$, or else we would have $a_y \in A(x) \subset A(z)$ for some z covering x as well as y , contradicting $x \vee y$ not covering x and y . Let $\gamma : \mathcal{A}(L) \rightarrow [n]$ be any atom ordering such that $\gamma(a_x) = 1$ and $\gamma(a_y) = 2$. We are now ready to use a similar argument to the one we gave in the special case where $x \wedge y = \hat{0}$ to show that the minimal labeling λ_γ given by γ is not an *EL*-labeling.

Since $a_x \in A(x) \setminus A(x \wedge y)$ and $\gamma(a_x) = 1$, we know that $\gamma(a_x) = \min\{\gamma(a) \mid a \in A(x) \setminus A(x \wedge y)\}$ and therefore $\lambda_\gamma(x \wedge y, x) = 1$. Then the lexicographically smallest chain in the interval $[x \wedge y, x \vee y]$ is of the form

$$C := \quad x \wedge y \lessdot x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = x \vee y.$$

By Lemma 6, $a_y \notin A(x_1)$, because x_1 covers x . Therefore, $\lambda_\gamma(x, x_1) \neq 2$. By Lemma 4 there is no repetition in the label sequence, implying $\lambda_\gamma(x, x_1) > 2$. For some $1 < j \leq k$, we must have $a_y \in A(x_j) \setminus A(x_{j-1})$. Now $1 \notin \{\gamma(a) \mid a \in A(x_j) \setminus A(x_{j-1})\}$ for any $j > 1$, so $\gamma(a_y) = 2 = \min\{\gamma(a) \mid a \in A(x_j) \setminus A(x_{j-1})\}$.

But $\min\{\gamma(a) \mid a \in A(x_1) \setminus A(x_0)\} \geq 3$, so $\lambda_\gamma(x, x_1) > \lambda_\gamma(x_{j-1}, x_j)$. This contradicts the fact that the lexicographically smallest chain in the interval $[x \wedge y, x \vee y]$ must be increasing for λ_γ to be an *EL*-labeling. Thus, whenever x and y cover $x \wedge y$, then $x \vee y$ covers both x and y . By Proposition 3, this means that L is a geometric lattice. \square

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